Analysis and Design of Beams for Bending

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Lecture Notes:
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Objective - Analysis and design of beams

Beams - structural members supporting loads at various points along the member

Transverse loadings of beams are classified as concentrated loads or distributed loads.

Applied loads result in internal forces consisting of a shear force (from the shear stress distribution) and a bending couple (from the normal stress distribution).

Normal stress is often the critical design criteria

\[ \sigma_x = -\frac{M_y}{I} \]
\[ \sigma_m = \frac{|M| c}{I} = \frac{|M|}{S} \]

Requires determination of the location and magnitude of largest bending moment.
Classification of Beam Supports

Statically Determinate Beams

- (a) Simply supported beam
- (b) Overhanging beam
- (c) Cantilever beam

Statically Indeterminate Beams

- (d) Continuous beam
- (e) Beam fixed at one end and simply supported at the other end
- (f) Fixed beam
• Determination of maximum normal and shearing stresses requires identification of maximum internal shear force and bending couple.

• Shear force and bending couple at a point are determined by passing a section through the beam and applying an equilibrium analysis on the beam portions on either side of the section.

• Sign conventions for shear forces $V$ and $V'$ and bending couples $M$ and $M'$

(a) Internal forces (positive shear and positive bending moment)
Sample Problem 5.1

SOLUTION:

- Treating the entire beam as a rigid body, determine the reaction forces.
- Section the beam at points near supports and load application points. Apply equilibrium analyses on resulting free-bodies to determine internal shear forces and bending couples.
- Identify the maximum shear and bending-moment from plots of their distributions.
- Apply the elastic flexure formulas to determine the corresponding maximum normal stress.
Sample Problem 5.1

SOLUTION:

- Treating the entire beam as a rigid body, determine the reaction forces
  from \( \sum F_y = 0 = \sum M_B : \ R_B = 46 \text{kN} \quad R_D = 14 \text{kN} \)

- Section the beam and apply equilibrium analyses on resulting free-bodies

  \[
  \sum F_y = 0 \quad -20 \text{kN} - V_1 = 0 \quad V_1 = -20 \text{kN}
  \]

  \[
  \sum M_1 = 0 \quad (20 \text{kN})(0 \text{m}) + M_1 = 0 \quad M_1 = 0
  \]

  \[
  \sum F_y = 0 \quad -20 \text{kN} - V_2 = 0 \quad V_2 = -20 \text{kN}
  \]

  \[
  \sum M_2 = 0 \quad (20 \text{kN})(2.5 \text{m}) + M_2 = 0 \quad M_2 = -50 \text{kN} \cdot \text{m}
  \]

  \[
  V_3 = +26 \text{kN} \quad M_3 = -50 \text{kN} \cdot \text{m}
  \]

  \[
  V_4 = +26 \text{kN} \quad M_4 = +28 \text{kN} \cdot \text{m}
  \]

  \[
  V_5 = -14 \text{kN} \quad M_5 = +28 \text{kN} \cdot \text{m}
  \]

  \[
  V_6 = -14 \text{kN} \quad M_6 = 0
  \]
Sample Problem 5.1

- Identify the maximum shear and bending-moment from plots of their distributions.

\[ V_m = 26 \text{kN} \quad M_m = |M_B| = 50 \text{kN} \cdot \text{m} \]

- Apply the elastic flexure formulas to determine the corresponding maximum normal stress.

\[ S = \frac{1}{6} b h^2 = \frac{1}{6} (0.080 \text{m})(0.250 \text{m})^2 \]
\[ = 833.33 \times 10^{-6} \text{m}^3 \]

\[ \sigma_m = \frac{|M_B|}{S} = \frac{50 \times 10^3 \text{N} \cdot \text{m}}{833.33 \times 10^{-6} \text{m}^3} \]

\[ \sigma_m = 60.0 \times 10^6 \text{Pa} \]
SOLUTION:

• Replace the 10 kip load with an equivalent force-couple system at $D$. Find the reactions at $B$ by considering the beam as a rigid body.

• Section the beam at points near the support and load application points. Apply equilibrium analyses on resulting free-bodies to determine internal shear forces and bending couples.

• Apply the elastic flexure formulas to determine the maximum normal stress to the left and right of point $D$.

The structure shown is constructed of a W10x112 rolled-steel beam. (a) Draw the shear and bending-moment diagrams for the beam and the given loading. (b) determine normal stress in sections just to the right and left of point $D$. 
Sample Problem 5.2

SOLUTION:

- Replace the 10 kip load with equivalent force-couple system at D. Find reactions at B.
- Section the beam and apply equilibrium analyses on resulting free-bodies.

From A to C:
\[ \sum F_y = 0 \quad -3x - V = 0 \quad V = -3x \text{ kips} \]
\[ \sum M_1 = 0 \quad (3x)(\frac{1}{2}x) + M = 0 \quad M = -1.5x^2 \text{ kip} \cdot \text{ft} \]

From C to D:
\[ \sum F_y = 0 \quad -24 - V = 0 \quad V = -24 \text{ kips} \]
\[ \sum M_2 = 0 \quad 24(x - 4) + M = 0 \quad M = (96 - 24x) \text{ kip} \cdot \text{ft} \]

From D to B:
\[ V = -34 \text{ kips} \quad M = (226 - 34x) \text{ kip} \cdot \text{ft} \]
Sample Problem 5.2

- Apply the elastic flexure formulas to determine the maximum normal stress to the left and right of point $D$.

From Appendix C for a W10x112 rolled steel shape, $S = 126$ in$^3$ about the $X$-$X$ axis.

To the left of $D$:

$$\sigma_m = \frac{|M|}{S} = \frac{2016 \text{kip} \cdot \text{in}}{126 \text{in}^3} = 16.0 \text{ksi}$$

To the right of $D$:

$$\sigma_m = \frac{|M|}{S} = \frac{1776 \text{kip} \cdot \text{in}}{126 \text{in}^3} = 14.1 \text{ksi}$$
• Relationship between load and shear:

\[ \sum F_y = 0 : \quad V - (V + \Delta V) - w \Delta x = 0 \]

\[ \Delta V = -w \Delta x \]

\[ \frac{dV}{dx} = -w \]

\[ V_D - V_C = - \int_{x_C}^{x_D} w \, dx \]

• Relationship between shear and bending moment:

\[ \sum M_{C'} = 0 : \quad (M + \Delta M) - M - V \Delta x + w \Delta x \frac{\Delta x}{2} = 0 \]

\[ \Delta M = V \Delta x - \frac{1}{2} w (\Delta x)^2 \]

\[ \frac{dM}{dx} = V \]

\[ M_D - M_C = \int_{x_C}^{x_D} V \, dx \]
Sample Problem 5.3

SOLUTION:

- Taking the entire beam as a free body, determine the reactions at $A$ and $D$.

- Apply the relationship between shear and load to develop the shear diagram.

- Apply the relationship between bending moment and shear to develop the bending moment diagram.

Draw the shear and bending moment diagrams for the beam and loading shown.
SOLUTION:

- Taking the entire beam as a free body, determine the reactions at $A$ and $D$.

$$\sum M_A = 0$$
$$0 = D(24 \text{ ft}) - (20 \text{ kips})(6 \text{ ft}) - (12 \text{ kips})(14 \text{ ft}) - (12 \text{ kips})(28 \text{ ft})$$
$$D = 26 \text{ kips}$$

$$\sum F_y = 0$$
$$0 = A_y - 20 \text{ kips} - 12 \text{ kips} + 26 \text{ kips} - 12 \text{ kips}$$
$$A_y = 18 \text{ kips}$$

- Apply the relationship between shear and load to develop the shear diagram.

$$\frac{dV}{dx} = -w \quad dV = -w \, dx$$

- zero slope between concentrated loads
- linear variation over uniform load segment
Sample Problem 5.3

- Apply the relationship between bending moment and shear to develop the bending moment diagram.

\[
\frac{dM}{dx} = V \quad dM = V \, dx
\]

- bending moment at A and E is zero
- bending moment variation between A, B, C and D is linear
- bending moment variation between D and E is quadratic
- net change in bending moment is equal to areas under shear distribution segments
- total of all bending moment changes across the beam should be zero
Sample Problem 5.5

SOLUTION:

- Taking the entire beam as a free body, determine the reactions at C.

- Apply the relationship between shear and load to develop the shear diagram.

- Apply the relationship between bending moment and shear to develop the bending moment diagram.

Draw the shear and bending moment diagrams for the beam and loading shown.
Sample Problem 5.5

SOLUTION:

• Taking the entire beam as a free body, determine the reactions at C.

\[ \Sigma F_y = 0 = -\frac{1}{2} w_0 a + R_C \quad R_C = \frac{1}{2} w_0 a \]

\[ \Sigma M_C = 0 = \frac{1}{2} w_0 a \left( L - \frac{a}{3} \right) + M_C \quad M_C = -\frac{1}{2} w_0 a \left( L - \frac{a}{3} \right) \]

Results from integration of the load and shear distributions should be equivalent.

• Apply the relationship between shear and load to develop the shear diagram.

\[ V_B - V_A = -\int_0^a w_0 \left( 1 - \frac{x}{a} \right) dx = -\left[ \frac{w_0 x - \frac{x^2}{2a}}{2} \right]_0^a \]

\[ V_B = -\frac{1}{2} w_0 a = -\text{(area under load curve)} \]

- No change in shear between B and C.
- Compatible with free body analysis
Sample Problem 5.5

- Apply the relationship between bending moment and shear to develop the bending moment diagram.

\[
M_B - M_A = \int_0^a \left(-w_0 \left(x - \frac{x^2}{2a}\right)\right) dx = \left[-w_0 \left(\frac{x^2}{2} - \frac{x^3}{6a}\right)\right]_0^a
\]

\[
M_B = -\frac{1}{3} w_0 a^2
\]

\[
M_B - M_C = \int_a^L (-\frac{1}{2} w_0 a) dx = -\frac{1}{2} w_0 a (L - a)
\]

\[
M_C = -\frac{1}{6} w_0 a (3L - a) = \frac{a w_0}{2} \left(L - \frac{a}{3}\right)
\]

Results at C are compatible with free-body analysis.